MAT 219 (lecture 7)

§7.5 Homogeneous Systems with Constant Coeff.

→ Also some of §7.3 Eigenvalues & Eigenvectors

Next: §7.6 Complex Eigenvalues

Begin with messy computation. (to be cleaned up later)

EX Solve

$$\left.\begin{aligned} x' &= -x + 2y \\ y' &= 2x + 2y \end{aligned}\right\} \rightarrow \underline{x}' = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \underline{x}$$

where $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

Recall Solution should look like

$$\underline{x} = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$$

where $\underbrace{\lambda_1 \underline{v}_1}_{(\underline{v}_1 e^{\lambda_1 t})'} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \underline{v}_1}_{\text{plug into } \begin{smallmatrix} -x+2y \\ 2x+2y \end{smallmatrix}}$ & $\lambda_2 \underline{v}_2 = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \underline{v}_2$

Step 1 Find λ_1 & λ_2 "eigenvalues"

$$\begin{bmatrix} \lambda x = -x + 2y \\ \lambda y = 2x + 2y \end{bmatrix} \rightsquigarrow \begin{bmatrix} (0 = (-1-\lambda)x + 2y)(2-\lambda) \\ -\,(0 = 2x + (2-\lambda)y)(2) \end{bmatrix}$$

$$0 = (-1-\lambda)(2-\lambda)x - 2\cdot 2x$$

$x \neq 0 \rightsquigarrow$
$$0 = (\lambda^2 - \lambda - 6)x$$

"characteristic equation" →
$$0 = \lambda^2 - \lambda - 6$$

$$0 = (\lambda - 3)(\lambda + 2)$$

$\lambda = 3, -2$ ← Eigenvalues.

e^{3t} e^{-2t} in solution of DE.

(like integrating factor: $\underline{e^{\int \lambda \, dt}}$)

more about this later...

Step 2 Find $\underline{v}_1$ & $\underline{v}_2$ "eigenvectors"

$\lambda = 3$ Plug into eigenvalue equations

$$\begin{bmatrix} \lambda x = -x + 2y \\ \lambda y = 2x + 2y \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3x = -x + 2y \\ 3y = 2x + 2y \end{bmatrix}$$

simplify $\rightsquigarrow \begin{bmatrix} 4x = 2y \\ y = 2x \end{bmatrix}$ same line (good)

Note: Solution has $\underline{c_1 \underline{v}_1} e^{\lambda_1 t}$

so eigenvector can be anything on line $c_1 \underline{v}_1$

→ Eigenvalue equations should ALWAYS give two copies of this line (for 2×2)

→ pick $x = 1 \rightsquigarrow y = 2x = 2$

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda = -2}$ Plug into eigenvalue equations

$$\begin{bmatrix} \lambda x = -x + 2y \\ \lambda y = 2x + 2y \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2x = -x + 2y \\ -2y = 2x + 2y \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} -x = 2y \\ -4y = 2x \end{bmatrix}$$ same line (good)

pick $x = 1 \rightsquigarrow y = -\frac{1}{2}$ $\quad \underline{v}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$?? $\times 2$

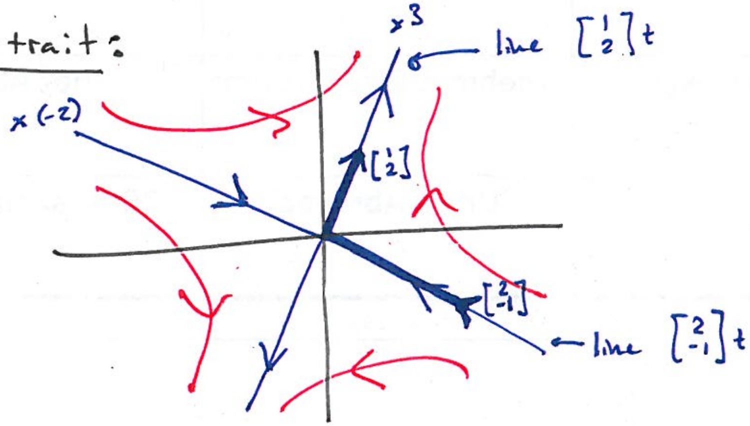
Better: pick $x = 2 \rightsquigarrow y = -1$ $\quad \underline{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Answer:

$$\underline{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-2t}$$

$$\left(\begin{matrix} x = c_1 e^{3t} + 2c_2 e^{-2t} \\ y = 2c_1 e^{3t} - c_2 e^{-2t} \end{matrix} \right)$$

Phase Portrait:



Let's clean this up.

Method to solve $\underline{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$

Step 1 Find eigenvalues

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{v} = \lambda \underline{v} \rightsquigarrow \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \underline{v} = \underline{0}$$

$$\begin{aligned} &((a-\lambda)x + by = 0)\,(d-\lambda) \\ -&(cx + (d-\lambda)y = 0)\,b \end{aligned}$$

$$(a-\lambda)(d-\lambda)x - bcx = 0$$

$$[(a-\lambda)(d-\lambda) - bc]\,x = 0$$

Characteristic equation:

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\left(\text{i.e. } \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0\right)$$

short-cut: $\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0$

Step 2 Find eigenvectors

plug λ into $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \underline{v} = \underline{0}$

→ Rows should be multiples!

$\underline{v}$ is $\perp$ to $\begin{bmatrix} a-\lambda \\ b \end{bmatrix} \rightsquigarrow \boxed{\begin{bmatrix} b \\ \lambda - a \end{bmatrix} = \underline{v}}$

slope $\frac{a-\lambda}{b}$ $\qquad$ slope $= -\frac{b}{a-\lambda}$

(you could also choose some other multiple)

EX: Solve
$$\underline{x}' = \begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix} \underline{x}$$

Eigenvalues:
$$\det \begin{bmatrix} 1-\lambda & -5 \\ 4 & -8-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-8-\lambda) + 20 = 0$$

$$\lambda^2 + 7\lambda + 12 = 0$$

trace $\begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix} = 1-8$

det $\begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix} = -8+20$

$$(\lambda+3)(\lambda+4) = 0$$

$$\lambda = -3, -4 \rightsquigarrow \left(e^{3t} \text{ \& } e^{4t} \text{ terms in solution}\right)$$

Eigenvectors:

negative reciprocal.

$\lambda = -3$
$$\begin{bmatrix} 1+3 & -5 \\ 4 & -8+3 \end{bmatrix} \underline{v} = 0 \rightsquigarrow \underline{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Note: rows equal

negative reciprocal

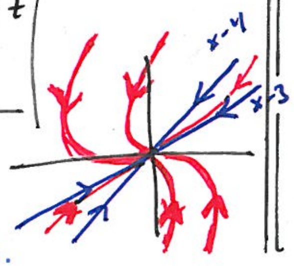
$\lambda = -4$
$$\begin{bmatrix} 1+4 & -5 \\ 4 & -8+4 \end{bmatrix} \underline{v} = 0 \rightsquigarrow \underline{v} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Note: rows equal

$\times 1/5$

(let's use $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ instead.)

$$\boxed{\text{Answer: } \underline{x} = c_1 \begin{bmatrix} 5 \\ 4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}}$$



→ When solutions all go in or out then they curve towards "louder" equilib. soln.

What about systems w/ >2 equations?

→ Same general method works. (with some modification...)

EX: Solve
$$\underline{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \underline{x}$$

Eigenvalues:
$$\det \begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(1-\lambda) + 2 + 2 - \left((1-\lambda) + 4(2-\lambda) + (1-\lambda)\right) = 0$$

Recall: det
$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

(If you reduce
$$(1-\lambda)x + y + 2z = 0$$
$$x + (2-\lambda)y + z = 0$$
$$2x + y + (1-\lambda)z = 0$$
to only x values by multiplying & subtr. then det(..) would be x coeff.)

$$-\lambda^3 + \underbrace{4}_{\text{trace}(A)}\lambda^2 - (2+2+1)\lambda + \underbrace{(6-10)}_{\det(A)} = 0$$

$$-(\lambda-1)(\lambda+1)(\lambda-4) = 0$$

$$\boxed{\lambda = -1, 1, 4}$$

Eigenvectors:

$\lambda = -1$

$$\begin{bmatrix} 1+1 & 1 & 2 \\ 1 & 2+1 & 1 \\ 2 & 1 & 1+1 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ \cancel{2} & \cancel{1} & \cancel{2} \end{bmatrix} \underline{v} = \underline{0}$$

One row is combination of other two.

(Can't do negative reciprocal)

$$2x + y + 2z = 0$$
$$x + 3y + z = 0$$

$\boxed{\text{Plan: Fix one variable and solve for others} \rightsquigarrow \text{usually take } x=1}$

pick $x = 1$

solve for y & z:

$$2 + y + 2z = 0$$
$$-(1 + 3y + z = 0)2$$
$$-5y = 0$$
$$y = 0 \rightsquigarrow 2 + 2z = 0$$
$$z = -1$$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

$$\boxed{\text{Check: } \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$\lambda = 1$

$$\begin{bmatrix} 1-1 & 1 & 2 \\ 1 & 2-1 & 1 \\ 2 & 1 & 1-1 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ \cancel{2} & \cancel{1} & \cancel{0} \end{bmatrix} \underline{v} = 0$$

(Ignore bottom row = 2x(middle row) − (top row))

pick $x = 1$

$$y + 2z = 0$$
$$-\quad 1 + y + z = 0$$
$$-1 \quad + z = 0$$
$$z = 1 \rightsquigarrow y =$$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

$$\boxed{\text{Check: } \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$\lambda = 4$

$$\begin{bmatrix} 1-4 & 1 & 2 \\ 1 & 2-4 & 1 \\ 2 & 1 & 1-4 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ \cancel{-2} & \cancel{1} & \cancel{-3} \end{bmatrix} \underline{v} = \underline{0}$$

(Ignore bottom row = (−1)(top row) + (−1)(middle))

pick $x = 1$

$$-3 + y + 2z = 0$$
$$-2(1 - 2y + z = 0)$$
$$-5 + 5y = 0$$
$$y = 1 \rightsquigarrow 1 - 2 + z = 0$$
$$z = 1$$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

actually... we could have guessed this:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\boxed{\text{Check: } \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Answer:

$$\underline{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$$

(Solve more problems if there is extra time)

Initial Value Problems & Fundamental Matrix

EX Solve

$$\underline{x}' = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \underline{x} \quad \text{with} \quad \underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

First we get the general solution:

Eigenvalues.

$$\det \begin{bmatrix} -2-\lambda & 6 \\ -2 & 5-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\boxed{\lambda = 1, \ 2}$$

Eigenvectors.

$\lambda = 1$

$$\begin{bmatrix} -2-1 & 6 \\ -2 & 5-1 \end{bmatrix} \underline{v} = 0$$

$$\underline{v} = \begin{bmatrix} 6 \\ -(-3) \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(check: $\begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

$\lambda = 2$

$$\begin{bmatrix} -2-2 & 6 \\ -2 & 5-2 \end{bmatrix} \underline{v} = \underline{0}$$

$$\underline{v} = \begin{bmatrix} 6 \\ -(-4) \end{bmatrix} \to \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(check: $\begin{bmatrix} -4 & 6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

General solution.

$$\underline{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t}$$

Note: Alternate Form of Solution

$$\underline{x} = \begin{bmatrix} 2e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \qquad \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t}$$

this is called the "Fundamental Matrix"

$$\Psi = \begin{bmatrix} 2e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$$

→ **Recall:** Wronskian is $\det(\Psi)$

Initial Values

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{x}(0) = \begin{bmatrix} 2\cancel{e^t} & 3\cancel{e^{2t}} \\ \cancel{e^t} & 2\cancel{e^{2t}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4-3} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Answer: $\underline{x} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t - \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t}$

Recall: Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Rightarrow \Psi^{-1} = \frac{1}{4e^{3t} - 3e^{3t}} \begin{bmatrix} 2e^{2t} & -3e^{2t} \\ -e^t & 2e^t \end{bmatrix}$$

Wronskian!